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A THEORY OF TIDAL MIXING IN AN ESSENTIALLY VERTICALLY HOMOGENEOUS ESTUARY

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GRADUATE DIVISION OF APPLIED MATHEMATICS

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Terminology

High tide;	the instant at which the water level in the estuary attains its maximum height;
Low tide;	the instant at which the water level in the estuary attains its minimum height;
Flood tide;	the time interval beginning at low tide and ending at high tide;
Ebb tide;	the time interval beginning at high tide and ending at low tide.

Introduction.

The distribution of salt or any other solute in an estuary is governed by the dynamic flow pattern and the extent to which portions of water having different solute concentrations mix with each other. The flow and the mixing are the result of the complex interactions of many factors, e.g., the flow due to the river, the tidal flow, the ground seepage, evaporation, wind stresses, gravity flow, etc. In many estuaries some of these factors are of much greater importance than others. In the following investigation we restrict ourselves to estuaries in which the tides and the river flow are the predominant agents, and we neglect all other influences.

We furthermore confine our attention to estuaries which are "essentially vertically homogeneous". By this we mean that if measurements are made of the concentration of the solute by taking samples over an entire transverse cross-section of the estuary (i.e., a vertical section which is perpendicular to the direction of the river flow) at any time during one or more tidal cycles (during which time the conditions of river flow and tidal amplitude are assumed to be unchanged), then the values of the concentration so obtained will differ from one another by a smaller order of magnitude than will the values obtained by taking measurements at various positions along the length of the estuary, but anywhere in a given cross-section and at any time. A typical case of an estuary which is not vertically homogeneous is one which exhibits two substantially distinct layers of different salinities, the more saline, and hence denser layer, being on the bottom, the fresher layer being on top. In such an estuary the differences in salinity between the two layers at any given position along the estuary and at any time will be greater than the differences between the salinities at points in the same layer but at various stations along the estuary. The division of estuaries into vertically homogeneous and "stratified" ones is, of course, not a sharp one, and cases which do not clearly fall into one or the other category will be encountered in practice.

The long range aim of a theoretical study of estuaries is to permit us to predict the distribution of any solute within the estuary from a knowledge of the river flow and the tides, possibly including the introduction into the estuary, or the removal therefrom, of any solute by means of an external source or sink, such as the discharge of a pollutant by an industrial plant. A complete theory would require a study of the detailed velocity distribution and of the mixing processes. At our present stage of knowledge the development of such a theory is much too difficult. We must content ourselves with a more macroscopic approach, i.e., one which aims only at predicting some convenient average variation of solute concentration as a function of position along the estuary and possibly of time. Such an approach has been made by Ketchum [1]* and by Arons and Stommel [2]. Ketchum subdivides the estuary along its axis into a number of volumes, each of which is as long as the average excursion of the tide in the neighborhood of the volume, and he assumes that complete mixing takes place within each volume at high tide.

* Numbers in square brackets refer to the Bibliography at the end of the paper.

Ketchum's method is essentially a refinement of an older method which assumes that all the water in the estuary mixes completely at high tide. Arons and Stommel formulated Ketchum's idea in terms of a diffusion equation which they assume to govern the solute distribution. The equation involves an "eddy diffusivity" whose value is assumed to depend on the amplitude of the tidal velocity and the tidal excursion.

The following investigation represents an attempt to derive (rather than to postulate) diffusion equations by assuming that the essence of the mixing process can be described in terms of very simple physical models. It is hoped that this approach will give new insight into the problem by showing clearly what the assumptions are that lead to a certain diffusion equation, and how modifications or generalizations of the assumptions change the equation. Moreover, such an approach may provide some guidance in the development of diffusion equations describing other physical models in which various mixing processes may be postulated. Solutions of these equations may then be compared with observation or experiment to determine in which estuaries, if any, the solute distribution is adequately described by one or another equation. In this way we may hope to make the initial steps in formulating laws which describe the solute diffusion in various estuaries and in beginning to understand the function of the various physical factors involved.

Excepting the final section of this report, it will always be assumed that all quantities are periodic in time with the period of a tidal cycle. In particular, this means that the time average of the solute distribution over a tidal cycle is the same for every tidal cycle.

For brevity we shall usually speak of the salinity, but it is to be understood that the investigation applies equally well to any solute.

We begin our considerations of mixing by examining an estuary or channel in which the motion of fluid and the distribution of salinity (or the distribution of any pollutant that is introduced into the estuary) will be assumed to be governed by three factors:

1) The tides, which are assumed here to cause a periodic displacement of part of the water of the estuary relative to the rest of the water and hence permit the possibility of the mixing of water from different parts of the estuary. (Thus, specifically, in these considerations in which we investigate the salinity distribution, this tidal motion permits a mixing of the more saline water downstream with that which is less saline further upstream.) Our basic picture will assume that the velocities due to the river are much smaller than those due to the tides so that a pronounced upstream flow occurs during flood tide.

2) A small scale mixing or eddy diffusion, distances moved by the fluid because of this mixing mechanism being assumed small relative to the distances which the fluid moves due directly to the tides. We shall not inquire into the detailed dynamics that cause this small scale mixing but will concern ourselves only with the effect of the mixing.

3) The river which produces a net flow of water (and hence also of salt) downstream. The transport of water due to the tides and subsequent small scale mixing will be found to produce a net flux of salt upstream by mixing the more saline water nearer the mouth of the estuary with that which is less saline further upstream. These two net fluxes will be set equal in treating the equilibrium condition in which there is no net flux of salt upstream over a tidal cycle.

The dynamics which are due to the tides and the effect of the small scale mixing as well as to the river are hypothesized. The salinity distribution then follows from the equations of continuity of salt and fluid, since we assume that the net flux of salt across any section of the estuary is zero during a tidal period. In the derivation of the salinity distribution we shall first find the flux that would be produced by the tides and small scale mixing in the absence of a river flow. When next taking the river flow into account we assume that the velocities which it causes are small in magnitude when compared with the velocities caused by the tides during most of the tidal cycle, and hence that the basic picture first described in the absence of river flow may be preserved when the river flow is included.

The channel is imagined to be divided as shown into two regions; a lower one of cross-sectional area $b_1(x,t)$, salinity $s_1(x,t)$ (dimensions ML^{-3}) in which the water moves with velocity $u_1(x,t)$ and an upper one with cross-sectional area, salinity and velocity $b_2(x,t)$, $s_2(x,t)$, $u_2(x,t)$ respectively; x being the distance coordinate along the length of the channel and u_1, u_2 being the velocities due to the tides.

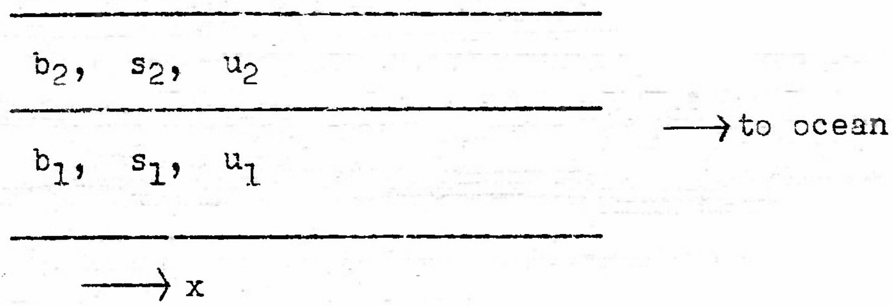


Fig. 1

Actually we need not envisage such a clear-cut division between the layers; they may be interspersed. We need imagine only that the channel contains fluid of two kinds, one with salinity s_1, moving with velocity u_1, the other with salinity s_2, moving with velocity u_2, and finally that the total cross-sectional area of the fluid of salinity s_1 be b_1, that of salinity s_2 be b_2. For convenience, however, we shall speak of the channel as if it contained two distinct layers.

Kinetic Considerations.

We begin by assuming the channel to be of constant width, but shall generalize later to include the case in which the width varies. For our consideration we shall need the equations of continuity for fluid and for salt and some assumptions about b_1 and b_2. Further, we will need to make some assumption regarding the transfer of salt between the two layers. We have, then, for the continuity of fluid in the lower layer,

$$\frac{\partial b_1}{\partial t} + \frac{\partial(b_1 u_1)}{\partial x} = 0 \tag{1}$$

and for continuity of fluid in the upper layer

$$\frac{\partial b_2}{\partial t} + \frac{\partial(b_2 u_2)}{\partial x} = 0. \tag{2}$$

Although we assume two distinct layers, each possessing its own velocity, an exchange of salt between the two layers is permitted, so that the equation for the continuity of salt is for the total channel:

$$\frac{\partial(b_1 s_1 + b_2 s_2)}{\partial t} + \frac{\partial(b_1 s_1 u_1 + b_2 s_2 u_2)}{\partial x} = 0. \qquad (3)$$

Substituting (1) and (2) in (3) we have

$$b_1 \left[\frac{\partial s_1}{\partial t} + u_1 \frac{\partial s_1}{\partial x}\right] + b_2 \left[\frac{\partial s_2}{\partial t} + u_2 \frac{\partial s_2}{\partial x}\right] = 0. \qquad (4)$$

It may be noted that in writing (1), (2), (3), and (4) we have assumed that the fluid in each layer moves with uniform velocity over the entire depth of the layer. If we assume instead that the fluid motion takes place over depths b_1', b_2' (i.e., not the entire depth of the layers) then (1), (2), (3), and (4) become, respectively,

$$\frac{\partial b_1}{\partial t} + \frac{\partial(b_1' u_1)}{\partial x} = 0 \qquad (1')$$

$$\frac{\partial b_2}{\partial t} + \frac{\partial(b_2' u_2)}{\partial x} = 0 \qquad (2')$$

$$\frac{\partial(b_1 s_1 + b_2 s_2)}{\partial t} + \frac{\partial(b_1' s_1 u_1 + b_2' s_2 u_2)}{\partial x} = 0 \qquad (3')$$

$$\left(b_1 \frac{\partial s_1}{\partial t} + b_1' u_1 \frac{\partial s_1}{\partial x}\right) + \left(b_2 \frac{\partial s_2}{\partial t} + b_2' u_2 \frac{\partial s_2}{\partial x}\right) = 0. \qquad (4')$$

In what follows we will assume, however, that $b_1' = b_1$, $b_2' = b_2$.

Before considering what laws might be formulated regarding the transfer of salt between the two layers we shall consider a specific simplified model which will help in the formulation of these laws and serve as a guide in the solution of the problem when we have more general models. We assume, for our first specific case, that

$$
\begin{aligned}
u_1 &= 0 \\
b_1 &= a \\
b_2 &= b + \eta \\
\eta &= \zeta \sin \omega t
\end{aligned}
\tag{5}
$$

where a, b, ζ and ω are positive constants. The equation $b_2 = b + \eta$ assumes that the tidal wave-length is large compared with the length of the estuary so that the level of the estuary rises uniformly over its length.

Thus we have a stationary lower layer of constant cross-sectional area and an upper layer the area of which varies sinusoidally with frequency ω, where $2\pi/\omega = \tau$ is the tidal period. Assuming that the fluid in the upper layer moves uniformly over the entire area $b_2 = b + \eta$, we may use the continuity equation (2) to determine u_2:

$$
\frac{\partial b_2}{\partial t} + \frac{\partial (b_2 u_2)}{\partial x} = \frac{\partial \eta}{\partial t} + \frac{\partial [(b + \eta) u_2]}{\partial x}
$$

$$
= \zeta \omega \cos \omega t + \frac{\partial [(b + \eta) u_2]}{\partial x} .
$$

Thus

$$
(b + \eta) u_2 = - \zeta x \omega \cos \omega t
$$

or

$$
u_2 = - \frac{\zeta x \omega \cos \omega t}{b + \zeta \sin \omega t} \tag{6}
$$

where we have set the arbitrary function of t which results from this integration equal to zero so that $u_2(0,t) = 0$ for all t. We keep in mind that the region of validity of the expressions for η and u_2 probably end for some positive value of x. Equation

(6) gives the velocity $u_2(x,t)$ of a particle in terms of its position x and the time t. The position of the particle, $\ell_2(t)$, as function of time may be found by substituting ℓ_2 for x and $\ell_2'(t)$ for u_2 in (6) and integrating:

$$\frac{d\ell_2}{dt} = - \frac{\zeta \ell_2 \omega \cos \omega t}{b + \zeta \sin \omega t}$$

or

$$\frac{d\ell_2}{\ell_2} = - \frac{\zeta\omega \cos \omega t}{b + \zeta \sin \omega t} dt$$

so that

$$\log \ell_2 = \log k \, (b + \zeta \sin \omega t)^{-1}$$

or

$$\ell_2 = \frac{k}{(b + \zeta \sin \omega t)} \tag{7}$$

where the constant of integration k may be determined by knowing the position at one particular time:

$$\ell_2(t_o) = \frac{k}{b + \zeta \sin \omega t_o}$$

from which

$$\ell_2(t) = \frac{\ell_2(t_o)(b + \zeta \sin \omega t_o)}{b + \zeta \sin \omega t} \, . \tag{7'}$$

Mixing Considerations -- Calculation of the Salt Flux.

Having considered the motion of the upper layer caused by the tides directly, we now consider the second factor which governs the salinity distribution, the small scale mixing which produces a transfer of salt between the two layers.

We assume that mixing takes place instantaneously at high and low tide only, and further that the mixing at each of these times is complete, i.e., that the salinities of the upper and lower layers are equal following each of these mixings. We shall discuss later the plausibility of this assumption. For the present we note that this assumption is in agreement with our concept of an essentially vertically homogeneous estuary.

Our object is to calculate the flux of salt past a given cross-section during a tidal period due to the action of the tides. Since our ultimate goal is to derive a differential equation for the salinity as a function of position along the estuary we shall want to describe the flux in terms of the salinity. The differential equation will express the fact that in an estuary in which conditions do not change from one tidal cycle to the next the upstream flux of salt due to the tides must be exactly balanced by the downstream flux due to the river. The flux will be obtained in the first instance (see Eq. (29)) from the mixing considerations described above in terms of the salinities at a fixed position but at different instants of time, namely those just prior to and immediately after the mixing at high and low tide. Using the equations of continuity of salt within the upper layer these salinities will then be expressed in terms of the salinities at a fixed time and at various x (see Eqs. (30), (33), and (34)). From these a differential equation for the salinity as a function of x is derived.

In the model we now consider the salinity of the upper layer, as measured in a frame of reference moving with the velocity of the upper layer, is constant, except when $\omega t = \pi/2$, $3\pi/2$, ... and hence

$$\frac{\partial s_2}{\partial t} + u_2 \frac{\partial s_2}{\partial x} = 0 \qquad (8)$$

except when $\omega t = \pi/2, 3\pi/2, \ldots$.

From (8), in particular,

$$s_2\left(x, \frac{(4n+1)\pi}{2\omega} - 0\right) = s_2\left(x + \xi_2, \frac{(4n-1)\pi}{2\omega} + 0\right), \qquad (9)$$

$$n = 0, \pm 1, \pm 2, \ldots$$

where, from (7'),

$$x + \xi_2 = \frac{x(b + \zeta)}{b - \zeta} . \qquad (10)$$

That is, the salinity of the upper layer at some point x along the estuary just before mixing at high tide is equal to the salinity of the upper layer just after the previous low tide mixing at the point $x + \xi_2$ further downstream, ξ_2 being the distance traveled from low to high tide by that section of the upper layer which is at x at high tide. In similar fashion

$$s_2\left(x, \frac{(4n+3)\pi}{2\omega} - 0\right) = s_2\left(x - \xi_1, \frac{(4n+1)\pi}{2\omega} + 0\right) \qquad (11)$$

$$n = 0, \pm 1, \pm 2, \ldots$$

where, from (7')

$$x - \xi_1 = \frac{x(b - \zeta)}{b + \zeta} , \qquad (12)$$

ξ_1 being the distance traveled from high to low tide by that section of the upper layer which is at x at low tide.

Since the lower layer is assumed to be stationary, we have

$$s_1\left(x, \frac{(4n+1)\pi}{2\omega} - 0\right) = s_1\left(x, \frac{(4n-1)\pi}{2\omega} + 0\right) \qquad (13)$$

$$s_1\left(x, \frac{(4n+3)\pi}{2\omega} - 0\right) = s_1\left(x, \frac{(4n+1)\pi}{2\omega} + 0\right), \qquad (14)$$

$$n = 0, \pm 1, \pm 2, \ldots .$$

Further, from the assumption of complete mixing after high and low tides, we have

$$s_1(x, \frac{(4n + 1)\pi}{2\omega} + 0) = s_2(x, \frac{(4n + 1)\pi}{2\omega} + 0) \qquad (15)$$

$$s_1(x, \frac{(4n + 3)\pi}{2\omega} + 0) = s_2(x, \frac{(4n + 3)\pi}{2\omega} + 0) \qquad (16)$$

$$n = 0, \pm 1, \pm 2, \ldots .$$

At this point we leave for a moment our rather formalized derivation and show how the flux over a tidal period may be derived quite simply by properly neglecting terms of relative order ζ/b. We shall not attempt to justify the particular approximations that are made therein since a more rigorous derivation will be given subsequently.

Derivation of Diffusion Equation from Elementary Physical Considerations.

Let us begin with the conditions just after high tide mixing and consider the transport of salt during a tidal cycle across the section at x. During ebb tide a volume of water of length ζ_1 (see Fig. 2) and cross-sectional area $b + \zeta \sim b$ moves downstream past x. Since no mixing takes place the volume carries all its salt content past the section x. At low tide it instantaneously mixes with the more saline water below thus acquiring a certain amount of salt. During flood tide the same volume of water (now of cross-sectional area $b - \zeta \sim b$ and length ζ_2) moves upstream across section x. At high tide this volume occupies the same position as it did at the beginning of our cycle, but, having acquired an amount of salt at low tide it has a higher salinity than it had just after high tide mixing. Moreover, the lower layer upstream of x, having lost some salt during low tide mixing with a section of the upper layer which had moved down during ebb from a position further upstream, is now less saline than at the beginning of our cycle. Hence, when mixing takes place at high tide, the upper layer gives up salt to the lower one. The cycle is now complete and we see that the net result has been an upstream transport or flux of a certain amount of salt past the section x. We now describe this process in mathematical terms.

Let

$$s_2\left(x, \frac{(4n + 1)\pi}{2\omega} + 0\right) = S_H(x) \tag{17}$$

$$s_2\left(x, \frac{(4n + 3)\pi}{2\omega} + 0\right) = S_L(x). \tag{18}$$

For

$$\zeta << b, \quad \xi_1 \sim \xi_2 \sim \int_{\pi/2\omega}^{3\pi/2\omega} u_2 dt \sim \int_{\pi/2\omega}^{3\pi/2\omega} - \frac{\zeta x\omega}{b} \cos \omega t \, dt = \frac{2x\zeta}{b}$$

from (6). Thus we may call

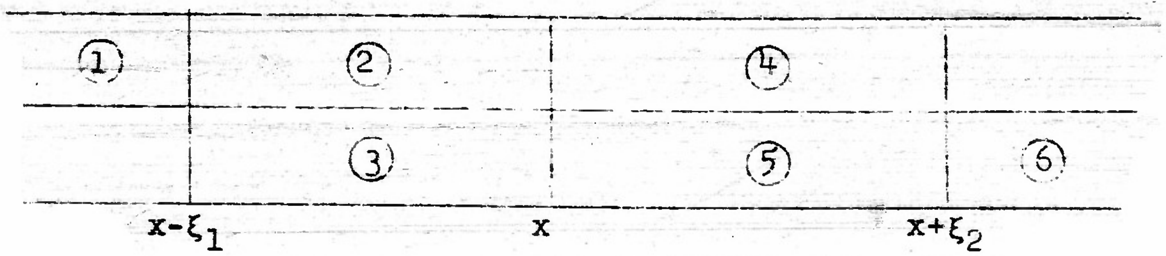
$$\xi = \frac{2x\zeta}{b} \tag{19}$$

the approximate amplitude of tidal displacement. (Note that if we neglect terms of higher order in ζ/b in (10) and (12), we have $\xi_1 \sim \xi_2 \sim \xi$.) Then, using (11), (14), (16), (17), and (18) we may write the equation for the continuity of salt at low tide as

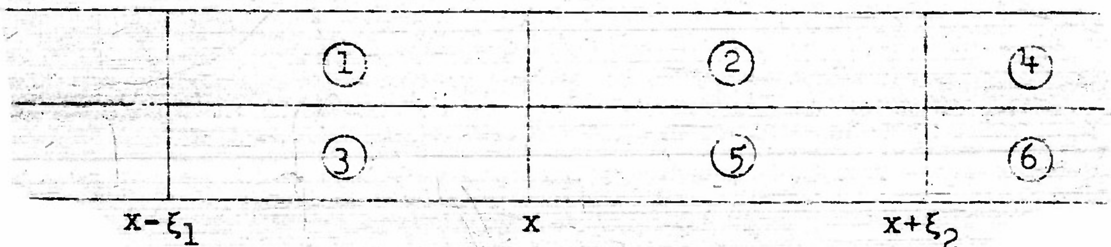
$$aS_H(x) + bS_H(x - \xi) = (a + b)S_L(x) \tag{20}$$

and that at high tide as

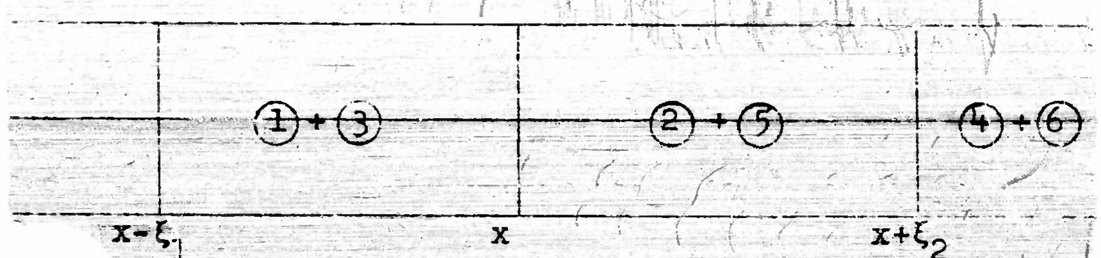
$$aS_L(x) + bS_L(x + \xi) = (a + b)S_H(x) \tag{21}$$



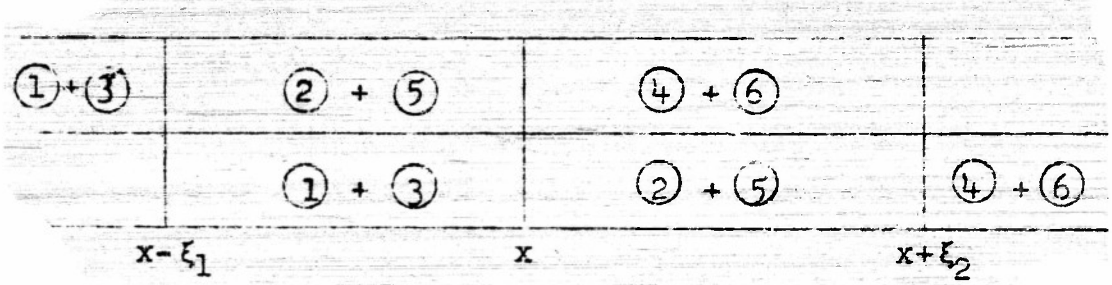
High tide after mixing $(t = \frac{\pi}{2\omega} + 0)$



Low tide before mixing $(t = \frac{3\pi}{2\omega} - 0)$



Low tide after mixing $(t = \frac{3\pi}{2\omega} + 0)$



High tide before mixing $(t = \frac{5\pi}{2\omega} - 0)$

Fig. 2

Schematic diagram showing motion of water due to tides and mixing.

in which we have substituted ξ for both ξ_1 and ξ_2 and neglected terms of relative order ζ/b (cf. (20) with (33)). Now the salt transport across a section due to the tides is approximately

$b\int_0^{\xi} S_L(x + \ell)d\ell \sim b\xi S_L(x + \frac{\xi}{2})$ up, during flood tide and

$b\int_0^{\xi} S_H(x - \ell)d\ell \sim b\xi S_H(x - \frac{\xi}{2})$ down, during ebb tide, so that the net flux up due to the tides during a tidal cycle across a section at x is

$$F = b\xi[S_L(x + \frac{\xi}{2}) - S_H(x - \frac{\xi}{2})] \quad . \tag{22}$$

Returning to (20) and (21) we may, for $\zeta/b << 1$, write these in the form

$$aS_H(x + \frac{\xi}{2}) + bS_H(x - \frac{\xi}{2}) = (a + b)S_L(x + \frac{\xi}{2}) \tag{20'}$$

$$aS_L(x - \frac{\xi}{2}) + bS_L(x + \frac{\xi}{2}) = (a + b)S_H(x - \frac{\xi}{2}) \tag{21'}$$

respectively. We may now substitute (20') in (22) and obtain an equation in S_H alone:

$$F = \frac{ab\xi}{a + b}[S_H(x + \frac{\xi}{2}) - S_H(x - \frac{\xi}{2})] \tag{23}$$

$$= \frac{ab\xi^2}{a + b} S_H'(x)$$

plus terms of higher order in ξ, and hence of higher order in ζ/b. This result is identical with that given later in (37) since from (19) $\xi = \frac{2x\zeta}{b}$ and since the constant c in (37) is finally set equal to zero. The difference in sign occurs because F is here the net flux upstream whereas in (37) F is the net flux downstream.

Similarly, substituting (21') in (22) we may obtain an equation in S_L alone:

$$F = \frac{ab\xi}{a+b}[S_L(x + \tfrac{\xi}{2}) - S_L(x - \tfrac{\xi}{2})] \tag{24}$$

$$= \frac{ab\xi^2}{a+b} S_L'(x)$$

plus terms of higher order in ξ. Thus if we call $\bar{s}(x) \equiv \frac{S_H(x) + S_L(x)}{2}$ the average salinity during a tidal cycle, then from (23) and (24)

$$F = \frac{ab\xi^2}{a+b} \bar{s}'(x). \tag{25}$$

Since we assume that there is no net increase of salt over a tidal cycle, this net flux upstream must equal the net flux downstream due to the river flow, so that

$$R_c \bar{s} = \frac{ab\xi^2}{a+b} \frac{d\bar{s}}{dx} \tag{26}$$

where R_c is the river flow per tidal cycle. Equation (26) is the integrated form of the diffusion equation

$$R_c \frac{d\bar{s}}{dx} = \frac{d}{dx}\left(\frac{ab\xi^2}{a+b} \frac{d\bar{s}}{dx}\right) \tag{27}$$

that is obtained again in (41). Using (19), Eq. (26) may be integrated, as is done in Eqs. (41) to (43).

Calculation of Salt Flux, continued.

Although this direct derivation of the flux and salinity is the most satisfactory as far as seeing the origin of the various terms in (26) is concerned, a more formal derivation in which the terms that are to be neglected are clearly exhibited was found preferable when generalizing the various factors that influence the final salinity. We turn therefore to the formal computation of the flux, $F(x)$, of salt at a fixed section x over a tidal period due to the tides and small scale mixing alone:

$$F(x) = \int_{\pi/2\omega - 0}^{5\pi/2\omega - 0} b_2 u_2 s_2 \, dt \qquad (28)$$

(F here is the net flux downstream.) In the integrand (which is a function of x and t) x is held constant when integrating over t. The choice of the limits of integration, $\pi/2\omega - 0$ to $5\pi/2\omega - 0$, is arbitrary; we could have chosen any interval of length $2\pi/\omega$. Since we want to speak of mixing or salt transfer between the two layers "at" high or low tide, i.e., when $\omega t = \frac{(4n+1)\pi}{2\omega}$ or $\omega t = \frac{(4n-1)}{2\omega}$ ($n = 0, \pm 1, \pm 2, \ldots$) it is convenient to have these points either definitely within or definitely outside of the range of integration, hence the "-0" following $\pi/2\omega$ and $5\pi/2\omega$ in the limits of integration. Since the integrand in (15) is finite for all t, we may omit the intervals $\pi/2\omega - 0 < t < \pi/2\omega + 0$ and $3\pi/2\omega - 0 < t < 3\pi/2\omega + 0$ and write

$$F(x) = \int_{\pi/2\omega+0}^{3\pi/2\omega-0} b_2 u_2 s_2 dt + \int_{3\pi/2\omega+0}^{5\pi/2\omega-0} b_2 u_2 s_2 dt. \qquad (28')$$

The reason for omitting the intervals within which mixing takes place will become apparent shortly. In (28') let $f(x,t) = b_2 u_2 s_2$ so that

$$\frac{dF}{dx} = \int_{\pi/2\omega+0}^{3\pi/2\omega-0} \frac{\partial f}{\partial x} dt + \int_{3\pi/2\omega+0}^{5\pi/2\omega-0} \frac{\partial f}{\partial x} dt. \qquad (28'')$$

Now

$$\frac{\partial f}{\partial x} = \frac{\partial(b_2u_2s_2)}{\partial x}$$

$$= s_2 \frac{\partial(b_2u_2)}{\partial x} + b_2u_2 \frac{\partial s_2}{\partial x}$$

$$= - s_2 \frac{\partial b_2}{\partial t} + b_2u_2 \frac{\partial s_2}{\partial x} \qquad \text{in view of (2)}$$

$$= b_2 \left(\frac{\partial s_2}{\partial t} + u_2 \frac{\partial s_2}{\partial x}\right) - \frac{\partial(b_2s_2)}{\partial t} .$$

However, in either of the intervals $\frac{\pi}{2\omega} + 0 \leq t \leq \frac{3\pi}{2\omega} - 0$, $\frac{3\pi}{2\omega} + 0 \leq t \leq \frac{5\pi}{2\omega} - 0$ we have $\frac{\partial s_2}{\partial t} + u_2 \frac{\partial s_2}{\partial x} = 0$, so that

$$\frac{dF}{dx} = - \int_{\pi/2\omega+0}^{3\pi/2\omega-0} \frac{\partial(b_2s_2)}{\partial t} dt - \int_{3\pi/2\omega+0}^{5\pi/2\omega-0} \frac{\partial(b_2s_2)}{\partial t} dt$$

$$= (b + \zeta)s_2(x, \frac{\pi}{2\omega} + 0) - (b - \zeta)s_2(x, \frac{3\pi}{2\omega} - 0) \qquad (29)$$

$$+ (b - \zeta)s_2(x, \frac{3\pi}{2\omega} + 0) - (b + \zeta)s_2(x, \frac{5\pi}{2\omega} - 0)$$

in view of (5).

We now express s_2 at $t = \frac{3\pi}{2\omega} - 0$, $\frac{3\pi}{2\omega} + 0$, and $\frac{5\pi}{2\omega} - 0$ in terms of s_2 at $t = \frac{\pi}{2\omega} + 0$. From (11)

$$s_2(x, \frac{3\pi}{2\omega} - 0) = s_2(x - \xi_1, \frac{\pi}{2\omega} + 0). \qquad (30)$$

In order to express s_2 at $\frac{3\pi}{2\omega} + 0$ and $\frac{5\pi}{2\omega} - 0$ in terms of s_2 at $\frac{\pi}{2\omega} + 0$ we need the equation of continuity of salt at the time of mixing, which may be written as

$$as_1(x, \frac{(4n + 1)\pi}{2\omega} - 0) + (b + \zeta)s_2(x, \frac{(4n + 1)\pi}{2\omega} - 0)$$
$$= as_1(x, \frac{(4n + 1)\pi}{2\omega} + 0) + (b + \zeta)s_2(x, \frac{(4n + 1)\pi}{2\omega} + 0) \qquad (31)$$

at high tide, and

$$as_1\left(x, \frac{(4n+3)\pi}{2\omega} - 0\right) + (b - \zeta)s_2\left(x, \frac{(4n+3)\pi}{2\omega} - 0\right)$$
$$= as_1\left(x, \frac{(4n+3)\pi}{2\omega} + 0\right) + (b - \zeta)s_2\left(x, \frac{(4n+3)\pi}{2\omega} + 0\right) \tag{32}$$

at low tide. Substituting (11), (14), and (16) in (32) we have

$$s_2\left(x, \frac{3\pi}{2\omega} + 0\right) =$$
$$\frac{1}{a + b - \zeta}\left[as_2\left(x, \frac{\pi}{2\omega} + 0\right) + (b - \zeta)s_2\left(x - \xi_1, \frac{\pi}{2\omega} + 0\right)\right] \tag{33}$$

and from (9) and (33) we have

$$s_2\left(x, \frac{5\pi}{2\omega} - 0\right) =$$
$$\frac{1}{a + b - \zeta}\left[as_2\left(x + \xi_2, \frac{\pi}{2\omega} + 0\right) + (b - \zeta)s_2\left(x, \frac{\pi}{2\omega} + 0\right)\right] \tag{34}$$

where it is to be noted that the argument of the term multiplying $(b - \zeta)$ in (34) is now the high tide position of that section of the upper layer which is found at $x + \xi_2$ at low tide, i.e., x, and not $x + \xi_2 - \xi_1$. Substituting (30), (33), and (34) in (29), we have, finally, substituting $\varepsilon = \zeta/b$

$$\frac{dF}{dx} = \frac{ab(1+\varepsilon)}{a + b(1-\varepsilon)}\left[s_2\left(x, \frac{\pi}{2\omega} + 0\right) - s_2\left(x + \xi_2, \frac{\pi}{2\omega} + 0\right)\right]$$
$$+ \frac{ab(1-\varepsilon)}{a + b(1-\varepsilon)}\left[s_2\left(x, \frac{\pi}{2\omega} + 0\right) - s_2\left(x - \xi_1, \frac{\pi}{2\omega} + 0\right)\right] \tag{35}$$

where, from (9') and from (10')

$$x + \xi_2 = \frac{x(1+\varepsilon)}{1-\varepsilon}$$
$$x - \xi_1 = \frac{x(1-\varepsilon)}{1+\varepsilon}. \tag{35'}$$

Assuming $\varepsilon << 1$ we expand the right hand side of (35') in a power series in ε and keep only the terms of lowest order in ε, which gives

$$\frac{dF}{dx} = -\frac{4\zeta^2}{b^2}\frac{ab}{a+b}\frac{\partial}{\partial x}\left\{x^2 \frac{\partial s_2(x, \frac{\pi}{2\omega}+0)}{\partial x}\right\}. \tag{36}$$

The flux of salt at a section at x, over the tidal period

$$\frac{\pi}{2\omega} - 0 \leq t \leq \frac{5\pi}{2\omega} - 0$$

is, therefore,

$$F = -\frac{4\zeta^2}{b^2}\frac{ab}{a+b}x^2\frac{\partial s_2(x, \frac{\pi}{2\omega}+0)}{\partial x} + C. \tag{37}$$

We now compute the flux due to the river flow (the third factor which determines the salt distribution) and add this flux to the flux produced by the tides (given in (37)), so that there is no net flux of salt over a tidal period. The flux due to the river is

$$G = \int_{\pi/2\omega-0}^{5\pi/2\omega-0} Rs^*(x,t)dt \tag{38}$$

where R is the river flow across any section and has the dimensions of L^3T^{-1}, and s^* is related to s_1 and s_2 as follows:

If the river produced a flow in each of the layers proportional to the cross-section of the layer, then we might define $s^* = \dfrac{as_1 + (b+\eta)s_2}{a+b+\eta}$. If the river acted over a cross-section $A(x,t)$ on the lower layer and over a cross-section $B(x,t)$ on the upper layer, then we would have $s^* = \dfrac{As_1 + Bs_2}{A+B}$. However, over a tidal cycle the net flux produced by the tides and small scale mixing and by the river flow is z so that the salinity of each of the layers is periodic. Thus we may write, for any time t, $s_2(x, \frac{3\pi}{2\omega}+0) \leq s_1(x,t) \leq s_2(x, \frac{\pi}{2\omega}+0)$, $s_2(x, \frac{5\pi}{2\omega}-0) \leq s_2(x,t) \leq s_2(x, \frac{3\pi}{2\omega}-0)$, which, together with (30), (33), (34), (9') and (12) give, for any time t,

$$s_1(x,t) = s_2(x, \frac{\pi}{2\omega} + 0) + O(\frac{\zeta}{b})$$

$$s_2(x,t) = s_2(x, \frac{\pi}{2\omega} + 0) + O(\frac{\zeta}{b}) \qquad (39)$$

$$s^*(x,t) = \frac{As_1 + Bs_2}{A + B} = s_2(x, \frac{\pi}{2\omega} + 0) + O(\frac{\zeta}{b}).$$

Thus (38) becomes

$$G = \frac{2\pi}{\omega} R \, [s_2(x, \frac{\pi}{2\omega} + 0) + O(\frac{\zeta}{b})\,]. \qquad (40)$$

Derivation of Salinity.

In adding the flux due to tides and small scale mixing to that due to the river flow we neglect the terms in (40) of $O(\frac{\zeta}{b})$. Having shown in (39) that the time variation of the salinity in either layer is of $O(\frac{\zeta}{b})$, we may again neglect terms of $O(\frac{\zeta}{b})$ and replace $s_2(x, \frac{\pi}{2\omega} + 0)$ in (37) and (40) by s(x), which may be interpreted to be the salinity at x to within terms of relative order $\frac{\zeta}{b}$ at any time. Adding F and G in (37) and (40) and setting their sum equal to zero we then have

$$\frac{4\zeta^2}{b^2} \frac{ab}{a + b} x^2 \frac{ds}{dx} - c = \frac{2\pi}{\omega} Rs, \qquad (41)$$

the solution of which is

$$\frac{\omega}{2\pi R} \log (\frac{2\pi R}{\omega} + c) = - \frac{1}{4 \frac{\zeta^2}{b^2} \frac{ab}{a + b} x} + c'$$

so that

$$s(x) = - \frac{\omega c}{2\pi R} + \frac{\omega}{2\pi R} e^{\frac{2\pi R}{\omega} c' - \frac{k}{x}} \qquad (41')$$

where

$$k = \frac{\pi}{2\omega} \frac{a + b}{ab} \frac{b^2}{\zeta^2} R. \tag{41"}$$

We now set $c = 0$ so that $s(0) = 0$, which is consistent with the fact that $u_2(0,t) =$ for all t so that no salt can get upstream of the section at $x = 0$.

The constant c' is determined by knowing the salinity at some particular point:

$$s(L) = \frac{\omega}{2\pi R} e^{\frac{2\pi R}{\omega} c' - \frac{k}{L}}$$

so that

$$\frac{\omega}{2\pi R} e^{\frac{2\pi R}{\omega} c'} = s(L) e^{\frac{k}{L}}$$

and hence

$$s(x) = s(L) e^{k(\frac{1}{L} - \frac{1}{x})} . \tag{42}$$

In particular, if we denote the ocean salinity by σ, and define the length L of the estuary to be the distance between the point at which $s(x) = 0$ and $s(x) = \sigma$, then

$$\frac{s(x)}{\sigma} = e^{\frac{k}{L}(1 - \frac{L}{x})} . \tag{43}$$

It should be noted that Arons and Stommel [2] arrive at this same result using dimensional analysis and call k/L the flushing number. In terms of the notation used here their flushing number equals $\frac{R(a + b)}{2B\zeta^2 \omega L}$ where B is a constant which remains undetermined in the paper of Arons and Stommel. It arises in their paper from the assumption that the eddy diffusivity A of the salt transfer equation $\frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} = \frac{\partial}{\partial x} (A \frac{\partial s}{\partial x})$, which they postulate as a basis for their considerations, is given by $A = 2B\xi_0 U_0$ where $2\xi_0$ is the total excursion, over a tidal period, of a particle due to the tides, U_0 is the amplitude of the tidal velocity and u is the river velocity. Equating their flushing number to k/L (see Eq. (41")), we have

$$B = \frac{a}{\pi b} \tag{44}$$

which relates B to the parameters of this paper.

Calculation of Flux for Generalized Mixing Process.

Let us now consider the various ways in which we might make the present model less restrictive, in order that it have a better chance of corresponding with reality, and investigate how the salinity distribution is affected by the modifications which are introduced.

We first consider a model in which the dynamics are those previously described but instead of assuming mixing to take place at high and low tide alone ($t = \frac{\pi}{2\omega}$ and $t = \frac{3\pi}{2\omega}$, respectively), we assume that there is mixing at $t = \frac{\pi}{2\omega}, \frac{\pi}{\omega}, \frac{3\pi}{2\omega}, \frac{2\pi}{\omega}$. Moreover, we shall not demand that the mixing be complete at each of these times, but rather only that at each mixing a certain quantity of fluid from the top layer exchanges places with an equal quantity of fluid from the bottom layer. Let Q(x,t) be the cross-sectional area of the section of fluid that is exchanged during mixing at time t, and position x along the estuary. For brevity let

$$\begin{aligned} s_1^- &= s_1(x,t-0) \qquad & s_1^+ &= s_1(x,t+0) \\ s_2^- &= s_2(x,t-0) \qquad & s_2^+ &= s_2(x,t+0). \end{aligned} \tag{45}$$

Then in view of the continuity of salt

$$(b_1 - Q)s_1^- + Qs_2^- = b_1 s_1^+ \tag{46}$$

which may be written as

$$- b_1(s_1^+ - s_1^-) = Q(s_1^- - s_2^-) \qquad (46')$$

and

$$(b_2 - Q)s_2^- + Qs_1^- = b_2 s_2^+ \qquad (47)$$

which may be written as

$$- b_2(s_2^+ - s_2^-) = Q(s_2^- - s_1^-). \qquad (47')$$

If the exchange of salt between the two layers occurs continuously rather than at discrete points in time, then (46') and (47') read, respectively,

$$- b_1\left(\frac{\partial s_1(x,t)}{\partial t} + u_1 \frac{\partial s_1(x,t)}{\partial x}\right) = Q[s_1(x,t) - s_2(x,t)] \qquad (46'')$$

$$- b_2\left(\frac{\partial s_2(x,t)}{\partial t} + u_2 \frac{\partial s_2(x,t)}{\partial x}\right) = Q[s_2(x,t) - s_1(x,t)] \qquad (47'')$$

i.e., in either case, the rate of discharge of salt from any one layer into the other is proportional to the difference of the salinities of the two layers. It may be noted that (46") and (47") satisfy Eq. (4) for the continuity of salt:

$$b_1\left(\frac{\partial s_1}{\partial t} + u_1 \frac{\partial s_1}{\partial x}\right) + b_2\left(\frac{\partial s_2}{\partial t} + u_2 \frac{\partial s_2}{\partial x}\right) = 0.$$

Reexamining the case in which we had complete mixing, it may be noted, substituting (15) in (31) and (16) in (32), that both of these equations may be expressed in our present notation by the equation $b_1 s_1^- + b_2 s_2^- = (b_1 + b_2)s_2^+$ from which $s_2^+ = \dfrac{b_1 s_1^- + b_2 s_2^-}{b_1 + b_2}$ so that

$$- b_2(s_2^+ - s_2^-) = - b_2\left[\frac{b_1 s_1^- + b_2 s_2^-}{b_1 + b_2}\right] + b_2 s_2^- = \frac{1}{\frac{1}{b_1} + \frac{1}{b_2}}[s_2^- - s_1^-] \qquad (48)$$

or, alternately, since both (31) and (32) may be written as

$$b_1 s_1^- + b_2 s_2^- = b_1 s_1^+ + b_2 s_2^+ \tag{49}$$

or

$$b_1(s_1^+ - s_1^-) + b_2(s_2^+ - s_2^-) = 0, \tag{49'}$$

we may also write (48) in the form:

$$- b_1(s_1^+ - s_1^-) = \frac{1}{\frac{1}{b_1} + \frac{1}{b_2}} [s_1^- - s_2^-] . \tag{48'}$$

Thus for complete mixing $Q = \frac{1}{\frac{1}{b_1} + \frac{1}{b_2}}$, from which we have both $Q < b_1$ and $Q < b_2$, as might be hoped.

In postulating a mixing that is less than complete, we shall assume

$$Q = \frac{\alpha(t)}{\frac{1}{b_1} + \frac{1}{b_2}} \tag{50}$$

where α will be permitted to be a periodic function of t (with period $2\pi/\omega$) independent of x, and $0 \leq \alpha \leq 1$. It is zero at times other than $n\pi/2\omega$, i.e., when there is no mixing.

In the case in which we assumed mixing at high and low tides only the flux was given by (28) and (28'). To obtain the equation for the present model which will be analogous to (28') we omit the intervals about the four points $n\pi/2\omega$, $n = 1,2,3,4$. Following the work previously done, we arrive at the equation which corresponds to (29) and reads

$$\begin{aligned} \frac{dF}{dx} &= (b + \zeta)s_2(x, \tfrac{\pi}{2\omega} + 0) - bs_2(x, \tfrac{\pi}{\omega} - 0) \\ &+ bs_2(x, \tfrac{\pi}{\omega} + 0) - (b - \zeta)s_2(x, \tfrac{3\pi}{2\omega} - 0) \\ &+ (b - \zeta)s_2(x, \tfrac{3\pi}{2\omega} + 0) - bs_2(x, \tfrac{2\pi}{\omega} - 0) \\ &+ bs_2(x, \tfrac{2\pi}{\omega} + 0) - (b + \zeta)s_2(x, \tfrac{5\pi}{2\omega} - 0). \end{aligned} \tag{51}$$

As before, we must express the salinities at $n\pi/2\omega + 0$ $(n = 2,3,4)$ and at $n\pi/2\omega - 0 (n = 2,3,4,5)$ in terms of the salinity at $\pi/2\omega + 0$; this time through the use of (7'), (46'), (47'), (49) and (50). We now proceed to do this. We have (for the reasons mentioned in connection with the derivation of (9) and (11)),

$$s_2(x, \frac{\pi}{\omega} - 0) = s_2(x - \Delta_1, \frac{\pi}{2\omega} + 0) \tag{52}$$

where, from (7')

$$x - \Delta_1 = \frac{xb}{b + \zeta}, \tag{53}$$

Δ_1 being the distance travelled from high to mid-tide by that section of the upper layer which is at x at mid-tide. Next, from (47') and (50) we have

$$-b[s_2(x, \frac{\pi}{\omega} + 0) - s_2(x, \frac{\pi}{\omega} - 0)] = \frac{\alpha_2}{\frac{1}{a} + \frac{1}{b}}[s_2(x, \frac{\pi}{\omega} - 0) - s_1(x, \frac{\pi}{\omega} - 0)] \tag{54}$$

where

$$\alpha_n = \alpha(t = \frac{n\pi}{2\omega}). \tag{54'}$$

Noting that

$$s_1(x, \frac{\pi}{\omega} - 0) = s_1(x, \frac{\pi}{2\omega} + 0) \tag{55}$$

for the reasons mentioned in connection with (13) and (14) we have, upon substitution of (52) and (55) in (54),

$$s_2(x, \frac{\pi}{\omega} + 0) = [1 - \frac{a\alpha_2}{a+b}]s_2(x - \Delta_1, \frac{\pi}{2\omega} + 0) + \frac{a\alpha_2}{a+b} s_1(x, \frac{\pi}{2\omega} + 0). \tag{56}$$

Further, similarly to (52), we have

$$s_2(x, \frac{3\pi}{2\omega} - 0) = s_2(x - \Delta_2, \frac{\pi}{\omega} + 0) \tag{57}$$

where, from (7'),

$$x - \Delta_2 = \frac{x(b - \zeta)}{b}. \tag{58}$$

The physical meaning of Δ_2 as well as all further Δ_i ($i = 1, \ldots, 8$) which will be defined below can be determined from Eq. (7').

Substituting (57) in (56) we have

$$s_2(x, \frac{3\pi}{2\omega} -0) = [1 - \frac{a\alpha_2}{a+b}]s_2(x-\Delta_3, \frac{\pi}{2\omega} +0)+ \frac{a\alpha_2}{a+b} s_1(x-\Delta_2, \frac{\pi}{2\omega} + 0) \tag{59}$$

where, from (7')

$$x - \Delta_3 = x \frac{(b - \zeta)}{b + \zeta}. \tag{60}$$

Next, from (47'), (50) and (54'),

$$- (b - \zeta)[s_2(x, \frac{3\pi}{2\omega} + 0) - s_2(x, \frac{3\pi}{2\omega} - 0)] =$$

$$= \frac{\alpha_3}{\frac{1}{a} + \frac{1}{b-\zeta}} [s_2(x, \frac{3\pi}{2\omega} - 0) - s_1(x, \frac{3\pi}{2\omega} - 0)]. \tag{61}$$

Now, similarly to (55) we have

$$s_1(x, \frac{3\pi}{2\omega} - 0) = s_1(x, \frac{\pi}{\omega} + 0) \tag{62}$$

and from (46') and (50)

$$- a[s_1(x, \frac{\pi}{\omega} + 0) -s_1(x, \frac{\pi}{\omega} - 0)] = \frac{\alpha_2}{\frac{1}{a} + \frac{1}{b}} [s_1(x, \frac{\pi}{\omega} - 0) - s_2(x, \frac{\pi}{\omega} - 0)] \tag{63}$$

which may be written as

$$s_1(x, \frac{\pi}{\omega} + 0) = [1 - \frac{b\alpha_2}{a+b}]s_1(x, \frac{\pi}{\omega} - 0) + \frac{b\alpha_2}{a+b} s_2(x, \frac{\pi}{\omega} - 0) \tag{63'}$$

$$= [1 - \frac{b\alpha_2}{a+b}]s_1(x, \frac{\pi}{2\omega} + 0) + \frac{b\alpha_2}{a+b} s_2(x - \Delta_1, \frac{\pi}{2\omega} + 0)$$

from (55) and (52). Thus from (63') and (62) we have

$$s_1(x, \frac{3\pi}{2\omega} - 0) = [1 - \frac{b\alpha_2}{a+b}]s_1(x, \frac{\pi}{2\omega} - 0) + \frac{b\alpha_2}{a+b} s_2(x - \Delta_1, \frac{\pi}{2\omega} + 0) \tag{64}$$

and substituting (64) and (59) in (61) we have

$$
\begin{aligned}
s_2\left(x, \frac{3\pi}{2\omega} + 0\right) = \left[1 - \frac{a\alpha_3}{a+b-\zeta}\right] & \left\{ \left[1 - \frac{a\alpha_2}{a+b}\right] s_2\left(x - \Delta_3, \frac{\pi}{2\omega} + 0\right) + \frac{a\alpha_2}{a+b} s_1\left(x - \Delta_2, \frac{\pi}{2\omega} + 0\right) \right\} \\
+ \frac{a\alpha_3}{a+b-\zeta} & \left\{ \left[1 - \frac{b\alpha_2}{a+b}\right] s_1\left(x, \frac{\pi}{2\omega} + 0\right) + \frac{b\alpha_2}{a+b} s_2\left(x - \Delta_1, \frac{\pi}{2\omega} + 0\right) \right\} .
\end{aligned}
\tag{65}
$$

Next, in analogy to (52), we have

$$
s_2\left(x, \frac{2\pi}{\omega} - 0\right) = s_2\left(x + \Delta_4, \frac{3\pi}{2\omega} + 0\right) \tag{66}
$$

where, from (7'),

$$
x + \Delta_4 = \frac{xb}{b - \zeta} . \tag{67}
$$

Substituting (66) in (65) we have, using (7'),

$$
\begin{aligned}
s_2\left(x, \frac{2\pi}{\omega} - 0\right) = \left[1 - \frac{a\alpha_3}{a+b-\zeta}\right] & \left\{ \left[1 - \frac{a\alpha_2}{a+b}\right] s_2\left(x - \Delta_1, \frac{\pi}{2\omega} + 0\right) + \frac{a\alpha_2}{a+b} s_1\left(x, \frac{\pi}{2\omega} + 0\right) \right\} \\
\frac{a\alpha_3}{a+b-\zeta} & \left\{ \left[1 - \frac{b\alpha_2}{a+b}\right] s_1\left(x + \Delta_4, \frac{\pi}{2\omega} + 0\right) + \frac{b\alpha_2}{a+b} s_2\left(x + \Delta_5, \frac{\pi}{2\omega} + 0\right) \right\}
\end{aligned}
\tag{68}
$$

where

$$
x + \Delta_5 = \frac{xb^2}{b^2 - \zeta^2} . \tag{69}
$$

Now, from (47'), (50) and (54'),

$$
- b\left[s_2\left(x, \frac{2\pi}{\omega} + 0\right) - s_2\left(x, \frac{2\pi}{\omega} - 0\right)\right] = \frac{\alpha_4}{\frac{1}{a} + \frac{1}{b}} \left[s_2\left(x, \frac{2\pi}{\omega} - 0\right) - s_1\left(x, \frac{2\pi}{\omega} - 0\right)\right] \tag{70}
$$

and analogously to (55),

$$s_1(x, \frac{2\pi}{\omega} - 0) = s_1(x, \frac{3\pi}{2\omega} + 0). \tag{71}$$

Further, from (46') and (50),

$$\begin{aligned} - a\, [s_1(x, \frac{3\pi}{2\omega} + 0) - s_1(x, \frac{3\pi}{2\omega} - 0)] = \\ \frac{\alpha_3}{\frac{1}{a} + \frac{1}{b-\zeta}} [s_1(x, \frac{3\pi}{2\omega} - 0) - s_2(x, \frac{3\pi}{2\omega} - 0)] \end{aligned} \tag{72}$$

from which

$$\begin{aligned} s_1(x, \frac{3\pi}{2\omega} + 0) = \\ [1 - \frac{(b-\zeta)\alpha_3}{a + b-\zeta}]\, s_1(x, \frac{3\pi}{2\omega} - 0) + \frac{(b-\zeta)\alpha_3}{a + b - \zeta} s_2(x, \frac{3\pi}{2\omega} - 0). \end{aligned} \tag{73}$$

Substituting (59) and (64) in (73) we have

$$\begin{aligned} s_1(x, \frac{3\pi}{2\omega} + 0) = {} & [1 - \frac{(b-\zeta)\alpha_3}{a+b-\zeta}] \left\{ \begin{aligned} & [1 - \frac{b\alpha_2}{a+b}]\, s_1(x, \frac{\pi}{2\omega} + 0) \\ & + \frac{b\alpha_2}{a+b}\, s_2(x - \Delta_1, \frac{\pi}{2\omega} + 0) \end{aligned} \right\} \\ & + \frac{(b-\zeta)\alpha_3}{a+b-\zeta} \left\{ \begin{aligned} & [1 - \frac{a\alpha_2}{a+b}]\, s_2(x - \Delta_3, \frac{\pi}{2\omega} + 0) \\ & + \frac{a\alpha_2}{a+b}\, s_1(x - \Delta_2, \frac{\pi}{2\omega} + 0) \end{aligned} \right\}. \end{aligned} \tag{74}$$

Thus, substituting (74) in the right hand side of (71) and the result of that substitution as well as (68) in (70), we have

$$
\begin{aligned}
s_2\left(x, \frac{2\pi}{\omega} + 0\right) = {} & \left[1 - \frac{a\alpha_4}{a+b}\right] \Bigg\{ \left[1 - \frac{a\alpha_3}{a+b-\zeta}\right] \Bigg\{ \left[1 - \frac{a\alpha_2}{a+b}\right] s_2\left(x-\Delta_1, \frac{\pi}{2\omega} + 0\right) + \frac{a\alpha_2}{a+b}\, s_1\left(x, \frac{\pi}{2\omega} + 0\right) \Bigg\} \\
& + \frac{a\alpha_3}{a+b-\zeta} \Bigg\{ \left[1 - \frac{b\alpha_2}{a+b}\right] s_1\left(x+\Delta_4, \frac{\pi}{2\omega} + 0\right) + \frac{b\alpha_2}{a+b}\, s_2\left(x+\Delta_5, \frac{\pi}{2\omega} + 0\right) \Bigg\} \Bigg\} \\
& + \frac{a\alpha_4}{a+b} \Bigg\{ \left[1 - \frac{(b-\zeta)\alpha_3}{a+b-\zeta}\right] \Bigg\{ \left[1 - \frac{b\alpha_2}{a+b}\right] s_1\left(x, \frac{\pi}{2\omega} + 0\right) + \frac{b\alpha_2}{a+b}\, s_2\left(x-\Delta_1, \frac{\pi}{2\omega} + 0\right) \Bigg\} \\
& + \frac{(b-\zeta)\alpha_3}{a+b-\zeta} \Bigg\{ \left[1 - \frac{a\alpha_2}{a+b}\right] s_2\left(x-\Delta_3, \frac{\pi}{2\omega} + 0\right) + \frac{a\alpha_2}{a+b}\, s_1\left(x-\Delta_2, \frac{\pi}{2\omega} + 0\right) \Bigg\} \Bigg\}
\end{aligned}
\tag{75}
$$

Finally, we have in analogy to (52)

$$
s_2\left(x, \frac{5\pi}{2\omega} - 0\right) = s_2\left(x + \Delta_6, \frac{2\pi}{\omega} + 0\right) \tag{76}
$$

where, from (7')

$$
x + \Delta_6 = \frac{x(b + \zeta)}{b}\,. \tag{77}
$$

Substituting (76) in (75) we have, using (7')

$$
\begin{aligned}
s_2\left(x, \frac{5\pi}{2\omega} - 0\right) = {} & \left[1 - \frac{a\alpha_4}{a+b}\right] \Bigg\{ \left[1 - \frac{a\alpha_3}{a+b-\zeta}\right] \Bigg\{ \left[1 - \frac{a\alpha_2}{a+b}\right] s_2\left(x, \frac{\pi}{2\omega} + 0\right) + \frac{a\alpha_2}{a+b}\, s_1\left(x+\Delta_6, \frac{\pi}{2\omega} + 0\right) \Bigg\} \\
& + \frac{a\alpha_3}{a+b-\zeta} \Bigg\{ \left[1 - \frac{b\alpha_2}{a+b}\right] s_1\left(x+\Delta_7, \frac{\pi}{2\omega} + 0\right) + \frac{b\alpha_2}{a+b}\, s_2\left(x+\Delta_4, \frac{\pi}{2\omega} + 0\right) \Bigg\} \Bigg\}
\end{aligned}
\tag{78}
$$

plus terms on following page

(78) cont.

$$+ \frac{a\alpha_4}{a+b} \left\{ \begin{aligned} &[1 - \frac{(b-\zeta)\alpha_3}{a+b-\zeta}] \left\{ \begin{aligned} &[1 - \frac{b\alpha_2}{a+b}] s_1(x+\Delta_6, \frac{\pi}{2\omega} + 0) \\ &+ \frac{b\alpha_2}{a+b} s_2(x, \frac{\pi}{2\omega} + 0) \end{aligned} \right\} \\ &+ \frac{(b-\zeta)\alpha_3}{a+b-\zeta} \left\{ \begin{aligned} &[1 - \frac{a\alpha_2}{a+b}] s_2(x-\Delta_2, \frac{\pi}{2\omega} + 0) \\ &+ \frac{a\alpha_2}{a+b} s_1(x-\Delta_8, \frac{\pi}{2\omega} + 0) \end{aligned} \right\} \end{aligned} \right\}$$

where

$$x + \Delta_7 = \frac{x(b + \zeta)}{b - \zeta} , \qquad x - \Delta_8 = \frac{x(b^2 - \zeta^2)}{b^2} . \tag{79}$$

We now substitute (52), (56), (59), (65), (68), (75), and (78) in (51) and, making use of (53), (58), (60), (67), (69), (77), and (79), expand the resulting expression for dF/dx in a power series in ζ/b, keeping only the terms of lowest order in ζ/b. From this procedure we obtain an expression involving both $s_1(x, \pi/2\omega + 0)$ and $s_2(x, \pi/2\omega + 0)$. In order to simplify this expression we shall make the additional assumption that the mixing at high tide is complete (i.e., that $\alpha_1 = 1$ and $s_2(x, \pi/2\omega + 0) = s_1(x, \pi/2\omega + 0)$). We then obtain

$$\frac{dF}{dx} = - \frac{\zeta^2}{b^2} \frac{abM}{a+b} \frac{\partial}{\partial x} [x^2 \frac{\partial s_2(x, \frac{\pi}{2\omega} + 0)}{\partial x}] \tag{80}$$

where

$$M = \alpha_2 + 4\alpha_3 + \alpha_4 - 2\alpha_2\alpha_3 - \alpha_2\alpha_4 - 2\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4 . \tag{81}$$

It may further be noted that if we assumed the mixing at low tide to be complete ($\alpha_3 = 1$) and that at high tide arbitrary then the above equation is changed only in that α_3 must be replaced by α_1 throughout. Comparing (80) with (36) we may note that these equations are identical except that the coefficient "4" in (36) is replaced by M which reduces to 4 if we let $\alpha_2 = \alpha_4 = 0$ and $\alpha_3 = 1$, as it should. It may also be noted that the above coefficient reduces to unity if we set $\alpha_2 = \alpha_4 = 1$, $\alpha_3 = 0$. This is as it should be, since in general the flux is proportional to the square of the total excursion of the upper layer during a tidal cycle, and, as far as the mixing process is concerned, the upper layer has effectively moved (if $\alpha_2 = \alpha_4 = 1$, $\alpha_3 = 0$) only half the distance moved in the first model. Following the work previously done in going from Eqs. (36) to (42), we may note that with this more general model we arrive at Eqs. (42) and (43) for the salinity, except that the parameter k in each of these equations must be replaced by

$$k' = \frac{2\pi(a + b)}{\omega \ ab} \frac{b^2 R}{\zeta^2 M} . \tag{82}$$

In (82) it should be noted that since $0 \leq \alpha_n \leq 1$, $n = 2,3,4$ we may write

$$\begin{aligned} M &= 2\alpha_3(1 - \alpha_2) + 2\alpha_3(1 - \alpha_4) + \alpha_2\alpha_3\alpha_4 + \alpha_2 + \alpha_4(1 - \alpha_2) \\ &\geq 0 \end{aligned} \tag{83}$$

since each of the five terms in the second expression is non-negative. Noting that we may also write

$$\begin{aligned} M &= 4 - \alpha_2 - \alpha_4 - 2(1 - \alpha_2)(1 - \alpha_3) - 2(1 - \alpha_4)(1 - \alpha_3) - (1-\alpha_3)\alpha_2\alpha_4 \\ &\leq 4, \end{aligned} \tag{84}$$

since each of the five terms following the 4 in the second expression is negative, we have from (83) and (84)

$$0 \leq M \leq 4$$

where, from (83) the left hand equality is satisfied only if $\alpha_2 = \alpha_3 = \alpha_4 = 0$ and the right hand equality is satisfied only if $\alpha_2 = \alpha_4 = 0$ and $\alpha_3 = 1$. If a and b can be obtained from observations then by matching the observed salinity with that predicted by this theoretical model, it should be possible to obtain an estimate of M and hence to obtain an idea of where during the tidal cycle most of the mixing takes place, large values of M corresponding to a state in which most of the mixing takes place at high and at low tide.

Extension of Analysis to Channel of Varying Cross-Section.

We now consider a different extension of our original model. As first assumed, we again postulate mixing to occur only at high and low tides, and that the mixing at each of these times is complete. The dynamics are essentially those originally described. However, the cross-sectional area of the stationary layer is now $a(x)$ in place of the constant a, the average area of the moving layer is $b(x)$, and the time varying part of the moving layer is $\zeta(x) \sin \omega t$ in place of $\zeta \sin \omega t$. Following the calculations connected with the first model, we see that Eq. (29) is changed only in that now $b = b(x)$, $\zeta = \zeta(x)$. Equation (30): $s_2(x, \frac{3\pi}{2\omega} - 0) = s_2(x - \xi_1, \frac{\pi}{2\omega} + 0)$ must be modified only in that ξ_1 is no longer given by (12) since Eq. (6), from which it was derived, must be modified as is shown in (88). With this modification implied, (33) must now read

$$s_2(x, \frac{3\pi}{2\omega} + 0) =$$

$$\frac{1}{a(x)+b(x)-\zeta(x)} [a(x)s_2(x, \frac{\pi}{2\omega} + 0)+(b(x) - \zeta(x))s_2(x - \xi_1, \frac{\pi}{2\omega} + 0)] \tag{85}$$

Equation (34) must be slightly modified, however, since substituting (9) in (85) (rather than in (33)) we have

$$s_2(x, \frac{5\pi}{2\omega} - 0) =$$

$$\frac{[a(x+\xi_2)s_2(x+\xi_2, \frac{\pi}{2\omega} +0)+(b(x+\xi_2)- \zeta(x+\xi_2))s_2(x, \frac{\pi}{2\omega} + 0)]}{a(x+\xi_2) + b(x+\xi_2)- \zeta(x+\xi_2)} \quad (86)$$

in which it must be recalled that ξ_2 is no longer given by (10). Substituting (30), (85) and (86) in (29) we now have

$$\frac{dF}{dx} = \frac{a(x)[b(x) + \zeta(x)]}{a(x)+b(x) - \zeta(x)} [s_2(x, \frac{\pi}{2\omega} + 0) - s_2(x - \xi_1, \frac{\pi}{2\omega} + 0)]$$
$$+ \frac{a(x+\xi_2)[b(x)+\zeta(x)]}{a(x+\xi_2)+b(x+\xi_2)-\zeta(x+\xi_2)}[s_2(x, \frac{\pi}{2\omega} +0)-s_2(x+\xi_2, \frac{\pi}{2\omega} + 0)]. \quad (87)$$

We must now express $\xi_1(x)$ and $\xi_2(x)$ in terms of $b(x)$ and $\zeta(x)$, and hence must recalculate Eq. (7) giving the position of a particle as a function of time for the case in which a, b, ζ are variable.

Returning to Eq. (2) for the continuity of fluid in the upper layer we have

$$\frac{\partial}{\partial x}[(b(x) + \zeta(x)\sin \omega t)u_2] = - \frac{\partial}{\partial t}[b(x) + \zeta(x)\sin \omega t]$$

$$= - \zeta(x)\omega \cos \omega t$$

which, upon integrating, gives

$$[b(x) + \zeta(x)\sin \omega t]u_2 = - \omega \cos \omega t \int_0^x \zeta(x')dx' \text{ *}$$

*See remark following Eq. (6).

or

$$u_2 = -\frac{\omega \cos \omega t \int_0^x \zeta(x')dx'}{b(x) + \zeta(x) \sin \omega t} . \qquad (88)$$

Following the procedure used in the derivation of (7') we write

$$\frac{d\ell}{dt} = -\frac{\omega \cos \omega t \int_0^{\ell} \zeta(x')dx'}{b(\ell) + \zeta(\ell) \sin \omega t}$$

which may be written in the form

$$\frac{d}{d\ell} [\sin \omega t \int_0^{\ell} \zeta(x')dx'] = - b(\ell) \qquad (89)$$

regarding t as a function of ℓ. From (89), recalling that $x + \xi_2$ is the low tide position of a particle which is at x at high tide, we have

$$[\sin \omega t \int_0^{\ell} \zeta(x')dx']_{\ell=x, t=\pi/2\omega}^{\ell=x+\xi_2, t=3\pi/2\omega} = - \int_x^{x+\xi_2} b(\ell)d\ell$$

or

$$\int_0^{x+\xi_2} \zeta(x')dx' + \int_0^x \zeta(x')dx' = \int_x^{x+\xi_2} b(\ell)d\ell . \qquad (90)$$

Similarly, we have

$$[\sin \omega t \int_0^{\ell} \zeta(x')dx']_{\ell=x-\xi_1, t=\pi/2\omega}^{\ell=x, t=3\pi/2\omega} = - \int_{x-\xi_1}^{x} b(\ell)d\ell$$

or

$$\int_0^{x-\xi_1} \zeta(x')dx' + \int_0^x \zeta(x')dx' = \int_{x-\xi_1}^{x} b(\ell)d\ell . \qquad (91)$$

In order to expand dF/dx as given in (87) in a power series as was done in the case in which ζ and b were constants, it will be convenient to write

$$\zeta(x) = \zeta\lambda(x)$$
$$b(x) = b\mu(x) \tag{92}$$

where ζ and b are constants, having the dimensions and order of magnitude of $\zeta(x)$ and $b(x)$, respectively. $\lambda(x)$ and $\mu(x)$ are thus non-dimensional and of order one. With these substitutions (90) and (91) become

$$\frac{\zeta}{b} = \frac{\int_x^{x+\xi_2} \mu(\ell)d\ell}{\int_0^{x+\xi_2} \lambda(x')dx' + \int_0^x \lambda(x')dx'} \tag{90'}$$

$$\frac{\zeta}{b} = \frac{\int_{x-\xi_1}^{x} \mu(\ell)d\ell}{\int_0^{x-\xi_1} \lambda(x')dx' + \int_0^x \lambda(x')dx'} . \tag{91'}$$

In (90') and (91') we now consider

$$\varepsilon = \frac{\zeta(x)}{b(x)} \tag{92'}$$

as a function of ξ_2 and ξ_1 respectively, from which we finally obtain power series in ε for ξ_2 and ξ_1. Thus from (90')

$$\varepsilon(0) = 0$$

$$\frac{d\varepsilon(0)}{d\xi_2} = \frac{\mu(x)}{2\int_0^x \lambda(x')dx'} \tag{93}$$

$$\frac{d^2\varepsilon(0)}{d\xi_2^2} = \frac{\mu'(x)\int_0^x \lambda(x')dx' - \mu(x)\,\lambda(x)}{2[\int_0^x \lambda(x')dx']^2}$$

from which, considering ξ_2 as a function of ε, we have

$$\xi_2(0) = 0$$

$$\frac{d\xi_2(0)}{d\varepsilon} = \frac{2\int_0^x \lambda(x')dx'}{\mu(x)} \tag{94}$$

$$\frac{d^2\xi_2(0)}{d\varepsilon^2} = -\frac{4\int_0^x \lambda(x')dx'}{\mu^3(x)} \left[\mu'(x)\int_0^x \lambda(x')dx' - \mu(x)\lambda(x)\right].$$

Similarly, from (91') we have

$$\varepsilon(0) = 0$$

$$\frac{d\varepsilon(0)}{d\xi_1} = \frac{\mu(x)}{2\int_0^x \lambda(x')dx'} \tag{95}$$

$$\frac{d^2\varepsilon(0)}{d\xi_1^2} = \frac{-\mu'(x)\int_0^x \lambda(x')dx' + \mu(x)\lambda(x)}{2\left[\int_0^x \lambda(x')dx'\right]^2}$$

from which, considering ξ_1 as a function of ε, we have

$$\xi_1(0) = 0$$

$$\frac{d\xi_1(0)}{d\varepsilon} = \frac{2\int_0^x \lambda(x')dx'}{\mu(x)} \tag{96}$$

$$\frac{d^2\xi_1(0)}{d\varepsilon^2} = \frac{4\int_0^x \lambda(x')dx'}{\mu^3(x)}\left[\mu'(x)\int_0^x \lambda(x')dx' - \mu(x)\lambda(x)\right].$$

Using (92) we may write (87) in the form

$$\frac{dF}{dx} = \frac{a(x)[\mu(x) - \varepsilon\lambda(x)]}{\frac{a(x)}{b} + \mu(x) - \varepsilon\lambda(x)} \left[s_2(x, \frac{\pi}{2\omega} + 0) - s_2(x - \xi_1, \frac{\pi}{2\omega} + 0)\right]$$

$$+ \frac{a(x+\xi_2)[\mu(x) + \varepsilon\lambda(x)]}{\frac{a(x+\xi_2)}{b} + \mu(x+\xi_2) - \varepsilon\lambda(x+\xi_2)} \left[s_2(x, \frac{\pi}{2\omega} + 0) - s_2(x + \xi_2, \frac{\pi}{2\omega} + 0)\right]. \tag{87'}$$

Using (87'), (94) and (96), we now expand dF/dx in a power series in ε, keeping only the terms of lowest order in ε and obtain

$$\frac{dF}{dx} = -4 \frac{\partial}{\partial x} \left[\frac{a(x) \cdot b(x)}{a(x)+b(x)} \left(\frac{\int_0^x \zeta(x')dx}{b(x)} \right)^2 \frac{\partial s_2(x, \frac{\pi}{2\omega} + 0)}{\partial x} \right] \tag{97}$$

in which we have substituted (92) and (92') in order to return to the variables $\zeta(x)$ and $b(x)$. Comparing (97) with (36) and following the methods used to arrive at (42) and (43), we see that the salinity formerly given by (42) is now expressed by the following similar equation:

$$s(x) = s(L)e^{\int_L^x k(x')dx'} \tag{98}$$

where

$$k(x) = \frac{\pi R}{2\omega} \frac{a(x) + b(x)}{a(x) \cdot b(x)} \left[\frac{b(x)}{\int_0^x \zeta(x')dx'} \right]^2 \tag{98'}$$

which is identical with (42) if $a(x)$, $b(x)$ and $\zeta(x)$ are constant.

<u>Extension of Analysis to More General Kinetic Conditions.</u>

Finally, we consider a model which is identical to the first model (a, b, ζ constant, complete mixing at high and low tide and no mixing at other times) except that the bottom layer is assumed to have a periodic velocity

$$u_1(t) = -U \cos(\omega t + \varphi). \tag{99}$$

For this case the flux formerly given by (28) is now

$$\begin{aligned} F(x) &= \int_{\pi/2\omega-0}^{5\pi/2\omega-0} (b_2u_2s_2 + b_1u_1s_1)dt \\ &= \int_{\pi/2\omega+0}^{3\pi/2\omega-0} f(x,t)dt + \int_{3\pi/2\omega+0}^{5\pi/2\omega-0} f(x,t)dt \end{aligned} \tag{100}$$

where

$$f(x,t) = b_2u_2s_2 + b_1u_1s_1, \quad b_2 = b + \zeta \sin \omega t, \quad b_1 = a$$

and u_2 is given by (6), and u_1 by (99).

Following the work done previously we now have, in place of (29),

$$\begin{aligned} \frac{dF}{dx} &= (b + \zeta)s_2(x, \frac{\pi}{2\omega} + 0) + a\, s_1(x, \frac{\pi}{2\omega} + 0) \\ &\quad - (b + \zeta)s_2(x, \frac{\pi}{2\omega} - 0) - a\, s_1(x, \frac{5\pi}{2\omega} - 0) \end{aligned} \tag{101}$$

since from the continuity Eq. (32),

$$(b_1s_1 + b_2s_2)\Big|_{3\pi/2\omega-0} = (b_1s_1 + b_2s_2)\Big|_{3\pi/2\omega+0}.$$

Since we have assumed mixing complete after high and low tides, (15) and (16) may still be used. The continuity equations (31) and (32) may also be applied to this model and since u_2 is still given by (6), Eqs. (9), (10), (11), and (12) may also be used. However, (13) and (14) must now be changed to (see Eqs. (9) and (11))

$$s_1(x, \frac{(4n+1)\pi}{2\omega} - 0) = s_1(x + \gamma_2, \frac{(4n-1)\pi}{2\omega} + 0) \qquad (102)$$

$$s_1(x, \frac{(4n+3)\pi}{2\omega} - 0) = s_1(x - \gamma_1, \frac{(4n+1)\pi}{2\omega} + 0) \qquad (103)$$

$$n = 0, \pm 1, \pm 2, \ldots$$

where

$$\gamma_2 = -\int_{(4n-1)\pi/2\omega}^{(4n+1)\pi/2\omega} u_1(t)dt = \frac{2U\cos\varphi}{\omega} \qquad (102')$$

$$\gamma_1 = \int_{(4n+1)\pi/2\omega}^{(4n+3)\pi/2\omega} u_1(t)dt = \frac{2U\cos\varphi}{\omega}. \qquad (103')$$

Since with $b_1 = a$, a constant, we have $\gamma_2 = \gamma_1$, we denote their common value by

$$\gamma = \frac{2U\cos\varphi}{\omega}, \qquad (104)$$

the distance traveled by the lower layer between high and low tide.

We now express the various terms in (101) in terms of $s_2(x, \frac{\pi}{2\omega} + 0)$.

From (15) $s_1(x, \frac{\pi}{2\omega} + 0) = s_2(x, \frac{\pi}{2\omega} + 0)$ and

from (16) $s_1(x, \frac{3\pi}{2\omega} + 0) = s_2(x, \frac{3\pi}{2\omega} + 0)$ and

from (11) $s_2(x, \frac{3\pi}{2\omega} - 0) = s_2(x - \xi_1, \frac{\pi}{2\omega} + 0)$

where $x - \xi_1$ is given in (12).

From (103), (15) and (104) we have

$$s_1(x, \frac{3\pi}{2\omega} - 0) = s_2(x - \gamma, \frac{\pi}{2\omega} + 0). \qquad (105)$$

From (16) $s_1(x, \frac{3\pi}{2\omega} + 0) = s_2(x, \frac{3\pi}{2\omega} + 0)$

and substituting (11), (105) and (16) in (32) we have

$$s_2(x, \frac{3\pi}{2\omega} + 0) = \frac{1}{a+b-\zeta} [a \; s_2(x-\gamma, \frac{\pi}{2\omega} +0)+(b-\zeta)s_2(x-\xi_1, \frac{\pi}{2\omega} +0)]. \tag{106}$$

From (9) and (106)

$$s_2(x, \frac{5\pi}{2\omega} - 0) = \frac{1}{a+b-\zeta} [a \; s_2(x+\delta_1, \frac{\pi}{2\omega} + 0)+(b-\zeta)s_2(x, \frac{\pi}{2\omega} + 0)] \tag{107}$$

where
$$\delta_1 = \xi_2 - \gamma \tag{108}$$

following the method used for evaluating γ.

Finally, from (102), (16) and (106)

$$s_1(x, \frac{5\pi}{2\omega} - 0) = \frac{1}{a+b-\zeta} [a \; s_2(x, \frac{\pi}{2\omega} + 0)+(b-\zeta)s_2(x-\delta_2, \frac{\pi}{2\omega} + 0)] \tag{109}$$

where, from (7')
$$x - \delta_2 = \frac{(x + \gamma)(b - \zeta)}{b + \zeta} . \tag{110}$$

Substituting (11), (15), (16), (105), (106), (107), and (109) in (101) we have

$$\begin{aligned} \frac{dF}{dx} = \frac{a(b+\zeta)}{a+b-\zeta} & [s_2(x, \frac{\pi}{2\omega} + 0) - s_2(x + \delta_1, \frac{\pi}{2\omega} + 0)] \\ + \frac{a(b-\zeta)}{a+b-\zeta} & [s_2(x, \frac{\pi}{2\omega} + 0) - s_2(x - \delta_2, \frac{\pi}{2\omega} + 0)] . \end{aligned} \tag{111}$$

If we assume γ to be of order ζ/b and that $\varepsilon = \zeta/b << 1$ we may write

$$\begin{aligned} \frac{dF}{dx} = \frac{ab(1+\varepsilon)}{a+b(1-\varepsilon)} & [s_2(x, \frac{\pi}{2\omega} + 0) - s_2(x + \delta_1, \frac{\pi}{2\omega} + 0)] \\ + \frac{ab(1-\varepsilon)}{a+b(1-\varepsilon)} & [s_2(x, \frac{\pi}{2\omega} + 0) - s_2(x - \delta_2, \frac{\pi}{2\omega} + 0)] \end{aligned} \tag{111'}$$

where from (10), (12), (108), and (110)

$$\delta_1 = \frac{2x\varepsilon}{1 - \varepsilon} - \gamma \tag{112}$$

$$\delta_2 = \frac{2x\varepsilon}{1+\varepsilon} - \gamma\left(\frac{1-\varepsilon}{1+\varepsilon}\right). \qquad (113)$$

Using (111'), (112), (113), and (104) we expand the right hand side of (111') in a power series in ε and keep only the terms of lowest order in $\varepsilon = \zeta/b$. The result is

$$\frac{dF}{dx} = -\frac{ab}{a+b}\frac{\partial}{\partial x}\left[\left(\frac{2x\zeta}{b} - \gamma\right)^2 \frac{\partial s_2(x, \frac{\pi}{2\omega} + 0)}{\partial x}\right]. \qquad (114)$$

This result is just what one might have expected since, to first order in ε, the distance traveled by the upper layer in going from low to high tide is, as was previously shown, $\xi = \frac{2x\zeta}{b}$, whereas the distance now traveled by the lower layer in this same time interval is $\gamma = \frac{2U\cos\varphi}{\omega}$; and hence the term which corresponds to the diffusion coefficient is again proportional to the square of the relative displacements of the top and bottom layers. Following the work previously done, we could also solve (114) and obtain the salinity. The result is

$$s(x) = s(L)e^{k\left(\frac{1}{L-x_o} - \frac{1}{x-x_o}\right)} \qquad (115)$$

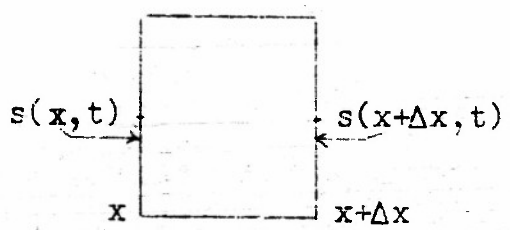
where

$$\frac{2x_o s}{b} = \gamma .$$

<u>Diffusion Equation in the Non-steady Case and in the Presence of a Solute Source Distribution.</u>

Let us return to equation (41). The left hand side expresses the flux of salt upstream past the cross-section at x during a tidal cycle as a consequence of the tides. The right hand side expresses the flux downstream past the same section and during the same interval due to the river. Equation (41) expresses the fact that these two contributions to the flux must cancel in an estuary in which conditions do not change from cycle to cycle and in which no salt (or other solute) is introduced into, or removed from, the estuary by external agents.

We now allow for variation of the solute concentration from cycle to cycle and also for introduction of solute into the estuary.



Consider an element of estuary of length Δx and of mean cross-sectional area during a tidal cycle a+b. The flux out of the element at x during the time Δt due to the tides is:

$$\frac{4\zeta^2}{b^2}\frac{ab}{a+b}\frac{\omega}{2\pi}x^2\frac{\partial s}{\partial x}\Delta t - \frac{c\omega}{2\pi}\Delta t.$$

The flux into the element at $x + \Delta x$ during time Δt due to the tides is (neglecting terms which will vanish in the limit $\Delta x,\ \Delta t \to 0$)

$$\frac{4\zeta^2}{b^2}\frac{ab}{a+b}\frac{\omega}{2\pi}\left[x^2\frac{\partial s}{\partial x} + \frac{\partial}{\partial x}\left(x^2\frac{\partial s}{\partial x}\right)\Delta x\right]\Delta t - \frac{c\omega}{2\pi}\Delta t.$$

The net flux into the element due to the tides during Δt is

$$\frac{4\zeta^2}{b^2}\frac{ab}{a+b}\frac{\omega}{2\pi}\frac{\partial}{\partial x}\left(x^2\frac{\partial s}{\partial x}\right)\Delta x\Delta t \qquad (116)$$

$$\equiv D\frac{\partial}{\partial x}\left(x^2\frac{\partial s}{\partial x}\right)\Delta x\,\Delta t.$$

Let us assume that the derivation of expression (116) remains valid if we permit changes in solute concentration from cycle to cycle as well as solute introduction by external agents (provided our fundamental mixing model is not altered). The time dependence of the salinity will most likely be due to variation in the river discharge. The external agent may be the discharge of pollution by an industry. The mixing model may then be expected to be essentially unchanged provided the time during which the river discharge varies significantly is large compared to a tidal cycle and provided the amount of solute which is introduced during a tidal cycle is so small that it does not materially affect the concentration during this cycle.

The flux into the element in time Δt at x due to the river is

$$R(t)s\Delta t.$$

The flux out of the element at $x + \Delta x$ due to the river is

$$R(s + \frac{\partial s}{\partial x}\Delta x)\Delta t.$$

The net flux out of the element due to the river is

$$R(t)\frac{\partial s}{\partial x}\Delta x\Delta t. \tag{117}$$

If $q(x,t)$ denotes the introduction of solute into the estuary per unit length of estuary and per unit time (dimensions $ML^{-1}T^{-1}$) then the total increase of solute in the element during Δt is

$$D\frac{\partial}{\partial x}(x^2\frac{\partial s}{\partial x})\Delta x\Delta t - R\frac{\partial s}{\partial x}\Delta x\Delta t + q(x,t)\Delta x\Delta t. \tag{118}$$

This must be equal to

$$[s(t + \Delta t) - s(t)](a + b)\Delta x$$

$$= \frac{\partial s}{\partial t}(a + b)\Delta x\Delta t. \tag{119}$$

Hence we obtain the following differential equation for the concentration of solute

$$(a+b)\frac{\partial s}{\partial t} + R(t)\frac{\partial s}{\partial x} = D\frac{\partial}{\partial x}\left(x^2\frac{\partial s}{\partial x}\right) + q(x,t). \qquad (119)$$

Conclusion.

In this report we have derived an equation governing the distribution of a solute in an estuary in which the river flow and the tides are the predominant factors and in which the dynamic and mixing conditions can be described by a very simple physical model. This model was subsequently generalized in various ways and it was found that the basic form of the diffusion equation was unchanged. It should be noted that the various generalizations which were discussed independently can all be combined into one model. In the last section an equation was derived which allows for time dependence of the various quantities involved and for introduction of solute into the estuary by an external agent.

In order to ascertain if the theory is capable of describing the conditions in a real estuary it will be necessary to compare the predictions of this theory with the observational data from a number of estuaries whose over-all structure permits application of this theory. In particular this will imply that the estuaries are essentially vertically homogeneous, that their properties are determined mainly by the river and tides and that it is feasible to divide the cross-section at any position along the estuary into two regions such that the tidal velocity is essentially uniform within each region but different for the two regions.

However, even if it is found that this theory is not directly applicable to real estuaries it may nevertheless be a helpful guide in experimental studies. Moreover, it is especially to be hoped that it will be a useful basis for the formulation of more elaborate theories that will describe real estuaries.

A few interesting solutions of the diffusion equation derived in the last section will be given in a future report.

Bibliography

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